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On Weyl's Gauge Field in a Non-Local Field

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Abstract

In this paper, a non-local field (i.e. the (x, ψ)-field) is constructed by regarding the spinor (ψ) as the internal freedom attached to each point (x). Since this field is likened to a unified field between the (x)- and (ψ)-fields, the metric is given by $d\sigma\psi = g_\lambda\, dx^\lambda \psi$. Concerning this, some conformally equivalent relations are considered. Next, Weyl's gauge field is introduced into the concept of connection in order to consider the gauge invariance. Finally, some essential features underlying our non-local field are grasped by formulating some fundamental equations of the spin curvature tensors.

1. *Introduction*

In this paper, we shall try to 'non-localise' our base field (i.e. the (x)-field) by attaching some internal structure to each point $(x) = (x^\kappa; \kappa = 1, 2, 3, 4)$ of the field. However, we shall here adopt the (four-component) spinor (ψ) as the internal freedom. Then, our non-local field, which will be called the (x, ψ)-field in the following, presents an aspect of a kind of unified field or interaction field between the (x)- and (ψ)-fields, the former is nothing more than a four-dimensional space-time manifold, whilst the latter is a physical field constituted by spinors. Or, more characteristically, the (x)- and (ψ)-fields might be likened to the gravitational field governed by general relativity and the spinor field governed by quantum mechanics respectively. This way of thinking descends from the ordinary unified field theory, and, as will be shown in the next section, our field theory is closely related to 'Wave Geometry' advanced by Mimura (1935).

Now, as is well known, the so-called non-local field theory was proposed by Yukawa (1950) for the purpose of resolving the divergence difficulty and finding the unified description of elementary particles. And we can say from the standpoint of geometrical theory of physical fields (Ikeda, 1972, 1973) that its essential way of thinking is to extend the point model of elementary particles to the non-point model. In other words, this means that the point (x) in the

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space-time manifold transforms to a certain internal structure, by which the (x)-field is 'non-localised'. We shall here adopt the spinor (ψ) as the internal degree of freedom in order to emphasise the non-vectorial property of it, as mentioned above. By the way, in the ordinary non-local field theory, a certain kind of directional vector is taken as the internal freedom, so that this is considered geometrically by Finsler geometry, as pointed out by Takano (1968).

In our non-local field, since the metric should embody the interaction of the (x)- and (ψ)-fields, it should be given by

$$d\sigma\psi = g_\lambda \, dx^\lambda \psi \tag{1.1}$$

where $d\sigma \equiv g_\lambda \, dx^\lambda$ plays the role of linear operator and g_λ represents the matrix-metric, which is assumed to contain the Dirac matrix γ_λ. We shall therefore consider in Section 2 some conformally equivalent relations associated with this metric and also pay attention to some relationship between our theory and wave geometry at the stage of metric.

In Section 3 we shall proceed to the concept of connection. We shall then apply Weyl's way of thinking (Weyl, 1918b) to our theory and introduce the gauge field in Weyl's style into the (x, ψ)-field. Concerning this gauge field, in Section 4 we shall attach importance to the pole which it plays at the stage of connection and also touch upon some essential features underlying this field by formulating some fundamental equations of the spin curvature tensors.

2. *Conformal Equivalence Associated with the Metric*

First, we shall consider the metric in the (x)-field. Since this field is the ordinary four-dimensional space-time manifold chosen as our base field, it is apposite that the metric in this field is given by

$$ds^2 = \gamma_{\lambda\kappa} \, dx^\kappa \, dx^\lambda \qquad (\kappa, \lambda = 1, 2, 3, 4) \tag{2.1}$$

where ds means the distance between two neighbouring points (x) and $(x + dx)$ and $\gamma_{\lambda\kappa}$ the metric tensor, the latter being stipulated by the Dirac matrix γ_λ as follows:

$$\gamma_{\lambda\kappa} = \gamma_{(\lambda}\gamma_{\kappa)} = (\gamma_\lambda\gamma_\kappa + \gamma_\kappa\gamma_\lambda)/2 \tag{2.2}$$

Next, we shall proceed to the metric in the (x, ψ)-field. As already emphasised, our non-local field should be regarded as a kind of interaction field between the (x)- and (ψ)-fields, so it is desirable that the metric in this field embodies the interaction. Then, against (2.1), we shall define the metric in the form

$$(d\omega^2 =)\, d\sigma\psi = g_\lambda \, dx^\lambda \psi \tag{2.3}$$

where $d\omega^2$ means the interaction coping with ds^2 and $d\sigma \equiv g_\lambda \, dx^\lambda$ the linearised arc length coping with $ds \equiv \gamma_\lambda \, dx^\lambda$, the latter being regarded as a linear operator operating on the state function ψ from a quantum mechanical point of view. Here, differentiating $d\sigma$ with respect to the proper time τ and supposing that the eigenvalue of $d\sigma/d\tau$ is the velocity of light c, we have the following equation:

$$(g_\lambda v^\lambda \psi =) g^\lambda v_\lambda \psi = c\psi \tag{2.4}$$

where $v^\lambda \equiv dx^\lambda/d\tau$ denotes the four-velocity. Then, adopting the natural units and replacing v_λ by the operator $[(1/mi)\nabla_\lambda]$, we obtain, after some manipulation, a generalised Dirac equation (for an electron) as follows (Yukawa *et al.*, 1972):

$$[ig^\lambda \nabla_\lambda + m]\psi = 0 \tag{2.5}$$

where m denotes the mass of an electron and ∇_λ the covariant derivative operator in the (x, ψ)-field, as defined in Section 4 (cf. (4.3)). This equation should also be regarded as a fundamental equation of the first order in the (x, ψ)-field (Ikeda, 1973; Takano, 1968).

The field theory, the metric of which is given by such a formula as (2.3), has been known as 'Wave Geometry' (Mimura, 1935; Mimura & Takeno, 1962; Mimura *et al.*, 1967). However, the metric of wave geometry is chosen as

$$ds\psi = \gamma_\lambda \, dx^\lambda \psi \tag{2.6}$$

Therefore, in order to consider the relationship between (2.3) and (2.6), we shall here introduce the matrix λ^κ. This can serve geometrically as the mapping operator between the (x)- and (ψ)-fields and can physically represent the interaction between them (Ikeda, 1972). g_λ and γ_λ are then combined with each other as follows:

$$g_\lambda = \lambda^\kappa \gamma_{\lambda\kappa} = \Lambda\gamma_\lambda \tag{2.7}$$

where we have put $\Lambda = \lambda^\kappa \gamma_\kappa$. This equation shows that there exists a conformally equivalent relation between g_λ and γ_λ through a scalar function Λ. At the same time we can obtain the conformally equivalent relations $d\sigma = \Lambda \, ds$ and $d\sigma\psi = \Lambda \, ds\psi$. Furthermore, if we put $g_{\lambda\kappa} = g_{(\lambda}g_{\kappa)}$, we have

$$g_{\lambda\kappa} = \Lambda^2 \gamma_{\lambda\kappa} \tag{2.8}$$

3. *On Weyl's Gauge Field*

First, we shall introduce the gauge (i.e. Eichung) in Weyl's style, in order to tie the conformal equivalence to Weyl's gauge invariance (Bregman, 1973; Utiyama, 1973; Weyl, 1918b). The Eichung $L(x)$ is nothing but the quadratic ground form determined at each point (x), which, in our case, is given by

$$L(x) = (g_\lambda \xi^\lambda \psi) \,.\, (g_\kappa \xi^\kappa \psi) \tag{3.1}$$

where ξ denotes an arbitrary vector annexed to the point (x) and the dot means the ordinary inner product in the tensor and spinor analyses. By virtue of (2.7) and (2.8) we can rewrite (3.1) as

$$L(x) = p(g_{\lambda\kappa} \xi^\kappa \xi^\lambda) = p\Lambda^2(\gamma_{\lambda\kappa} \xi^\kappa \xi^\lambda) \tag{3.2}$$

where we have put $p = \psi \,.\, \psi$. Equation (3.2) shows that the Eichung $L(x)$ is conformally equivalent to both $(g_{\lambda\kappa}\xi^\kappa\xi^\lambda)$ and $(\gamma_{\lambda\kappa}\xi^\kappa\xi^\lambda)$.

Next, we shall consider the gauge field and gauge invariance (i.e. Eichinvarianz). The essential point, in Weyl's way of thinking, is closely related to his 'Infinitesimalgeometrie' (Weyl, 1918a), which is grasped by the infinitesimal parallel displacement of the Eichung $L(x)$. Namely, according to Weyl 1918a, 1918b), the change (dL) of $L(x)$ due to the infinitesimal parallel displacement of the vector $\xi(x)$ from the point (x) to its neighbouring point $(x + dx)$ is assumed to be given by

$$dL = -(d\rho)L \tag{3.3}$$

where the differential form $d\rho$ is given by

$$d\rho = \rho_\mu \, dx^\mu \tag{3.4}$$

This ρ_μ constitutes the gauge field in Weyl's style, but in our case this may mean a general material field instead of the electromagnetic potential. At this time, the change $(d\xi)$ of ξ due to this parallel displacement is assumed to be given by

$$d\xi^\kappa = -\Gamma^\kappa_{\mu\lambda}\xi^\lambda \, dx^\mu \tag{3.5}$$

where $\Gamma^\kappa_{\mu\lambda}$ denotes the coefficient of connection in the (x, ψ)-field. Substituting (3.2) into (3.3) and taking account of (3.4) and (3.5), we then obtain the following relation:

$$(\partial_\mu + \rho_\mu + \partial_\mu \log p)g_{\lambda\kappa} = \Gamma^\nu_{\mu\lambda}g_{\nu\kappa} + \Gamma^\nu_{\mu\kappa}g_{\lambda\nu} \tag{3.6}$$

Also, under the assumption $\Gamma^\kappa_{\mu\lambda} = \Gamma^\kappa_{\lambda\mu}$, we can determine $\Gamma^\kappa_{\mu\lambda}$ uniquely as follows:

$$\Gamma^\kappa_{\mu\lambda} = \begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix} + \{\delta_\lambda{}^\kappa(\rho_\mu + \partial_\mu \log p) + \delta_\mu{}^\kappa(\rho_\lambda + \partial_\lambda \log p)$$
$$- g^{\kappa\nu}g_{\mu\lambda}(\rho_\nu + \partial_\nu \log p)\}/2 \tag{3.7}$$

where $\{{}^{\kappa}_{\mu\lambda}\}$ denotes the Christoffel three-index symbol of the second kind derived from $g_{\lambda\kappa}$ (Schouten, 1954), i.e.

$$\begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix} = g^{\kappa\nu}(\partial_\mu g_{\lambda\nu} + \partial_\lambda g_{\nu\mu} - \partial_\nu g_{\mu\lambda})/2 \tag{3.8}$$

Equation (3.6) shows that at the stage of connection, the 'protrusion' of our non-local field from Riemann space is caused by $(\rho_\mu + \partial_\mu \log p)$.

By the way, Weyl's original Eichung (Weyl, 1918b) is given by

$$l = \gamma_{\lambda\kappa}\xi^\kappa\xi^\lambda \tag{3.9}$$

which appears in (3.2). And Weyl's Eichinvarianz is expressed in the form

$$dl = -(d\phi)l \tag{3.10}$$

where $d\phi$ is prescribed by, as in (3.4),

$$d\phi = \phi_\mu \, dx^\mu \tag{3.11}$$

where ϕ_μ means the electromagnetic potential. Fortunately, in our case, we can obtain from (3.2)

$$L = (p\Lambda^2)l \tag{3.12}$$

Substituting this into (3.3) and taking account of (3.4), (3.10) and (3.11), we get the following relation:

$$\rho_\mu = \phi_\mu - \partial_\mu \log p - \partial_\mu \log \Lambda^2 \tag{3.13}$$

Thus, we can say that equation (3.12) gives the gauge transformation between L and l at the stage of Eichung, and that (3.13) gives the gauge transformation between ρ_μ and ϕ_μ under the requirement of Eichinvarianz.

4. *Some Fundamental Features Underlying the Non-Local Field*

In this section we shall consider some essential features of the (x, ψ)-field by formulating some fundamental equations of the spin curvature tensors. For this purpose we shall at first define the covariant derivative operator (∇_μ) which can preserve the gauge invariance. Hereupon, we can regard (3.6) as the covariant derivative of $g_{\lambda\kappa}$ and can rewrite it in the form

$$\nabla_\mu g_{\lambda\kappa} = D_\mu g_{\lambda\kappa} - \Gamma^\nu_{\mu\lambda} g_{\nu\kappa} - \Gamma^\nu_{\mu\kappa} g_{\lambda\nu} = 0 \tag{4.1}$$

where we have put $D_\mu = \partial_\mu + \rho_\mu + \partial_\mu \log p$. We can thus define the covariant derivatives of g_λ and ψ, respectively, as follows:

$$\nabla_\mu g_\lambda = D_\mu g_\lambda - \Gamma^\nu_{\mu\lambda} g_\nu - \Gamma_\mu g_\lambda + g_\lambda \Gamma_\mu \qquad (= 0) \tag{4.2}$$

and

$$\nabla_\mu \psi = D_\mu \psi - \Gamma_\mu \psi \tag{4.3}$$

where Γ_μ denotes the spin affine connection in the (x, ψ)-field (Takano, 1968). This covariant derivative operator (∇_μ) has already been used in such an equation as (2.5).

On the other hand, we can determine Γ_μ from (4.2) as follows:

$$\Gamma_\mu = A_{\mu\lambda\kappa} S^{\lambda\kappa}/4 + a_\mu I \tag{4.4}$$

where we have put

$$A^\nu_{\mu\lambda} g_\nu = \Gamma_\mu g_\lambda - g_\lambda \Gamma_\mu \tag{4.5}$$

$$S^{\lambda\kappa} = g^{[\lambda} g^{\kappa]} = (g^\lambda g^\kappa - g^\kappa g^\lambda)/2 \tag{4.6}$$

and $a_\mu = \mathrm{tr}(\Gamma_\mu)/4$, and I denotes the unit matrix (Bregman, 1973; Takano, 1968). In this case, $A_{\mu\lambda\kappa}$ constitutes the spin gauge field (Bregman, 1973) and a_μ represents an arbitrariness associated with Γ_μ, the latter has been interpreted as the electromagnetic potential (ϕ_μ) in Weyl's style in the ordinary spinor analysis (Bade & Jehle, 1953; Infeld & van der Waerden, 1933). But in our case, this arbitrariness is compensated by our gauge field (ρ_μ), as will be considered in the following.

For this purpose we must take notice of the arbitrariness associated with the spin curvature tensor ($P_{\mu\nu}$) (cf. (4.15)), but first we must consider some fundamental equations of the spin curvature tensors. The latter are then defined through the following relation:

$$2\nabla_{[\nu}\nabla_{\mu]}\psi = (P_{\nu\mu} + F_{\nu\mu})\psi \tag{4.7}$$

where we have put

$$P_{\nu\mu} = 2(D_{[\mu}\Gamma_{\nu]} + \Gamma_{[\nu}\Gamma_{\mu]}) \tag{4.8}$$

and

$$F_{\nu\mu} = 2D_{[\nu}D_{\mu]} \tag{4.9}$$

In this case $P_{\nu\mu}$ gives the ordinary spin curvature tensor (Takano, 1968), while $F_{\nu\mu}$ represents the curvature tensor derived essentially from the gauge field, which might be regarded as a generalised form of the fundamental tensor in the electromagnetic field, i.e. $f_{\nu\mu} = 2\,\partial_{[\nu}\phi_{\mu]}$. On the other hand, the other two curvature tensors are also introduced through the following relation:

$$2\nabla_{[\nu}\nabla_{\mu]}g_\lambda = R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}g_\kappa + Q_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}g_\kappa + F_{\nu\mu}g_\lambda \qquad (=0) \tag{4.10}$$

where we have put

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2(D_{[\mu}\Gamma_{\nu]\lambda}^{\kappa} + \Gamma_{[\nu|\lambda|}^{\iota}\Gamma_{\mu]\iota}^{\kappa} + \Gamma_{[\nu|\lambda|}^{\iota}A_{\mu]\iota}^{\kappa}) \tag{4.11}$$

and

$$Q_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2(D_{[\mu}A_{\nu]\lambda}^{\kappa} + A_{[\nu|\lambda|}^{\iota}A_{\mu]\iota}^{\kappa} + A_{[\nu|\lambda|}^{\iota}\Gamma_{\mu]}^{\kappa}{}_{\iota}) \tag{4.12}$$

In this case, $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ means the Riemann-Christoffel curvature tensor derived from $\Gamma_{\mu\lambda}^{\kappa}$ (Schouten, 1954) and $Q_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ the spin curvature tensor derived from $A_{\mu\lambda}^{\kappa}$, the latter is related to $P_{\nu\mu}$ as follows:

$$Q_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}g_\kappa = P_{\nu\mu}g_\lambda - g_\lambda P_{\nu\mu} \tag{4.13}$$

Now, since the equation (4.10) becomes zero identically in our case, we can obtain the following fundamental equation of the spin curvature tensors:

$$\begin{aligned} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}g_\kappa &= Q_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}g_\kappa + F_{\mu\nu}g_\lambda \\ &= P_{\mu\nu}g_\lambda - g_\lambda P_{\mu\nu} + F_{\mu\nu}g_\lambda \end{aligned} \tag{4.14}$$

where we have used (4.13). Therefore, we can, as in the same way as (4.4), determine the spin curvature tensor $P_{\mu\nu}$ as follows:

$$P_{\mu\nu} = R_{\nu\mu\lambda\kappa}S^{\lambda\kappa}/4 - D_{[\mu}D_{\nu]} + b_{\mu\nu}I \tag{4.15}$$

where $b_{\mu\nu} = \mathrm{tr}(P_{\mu\nu})/4$ represents the arbitrariness. However, by virtue of (4.4), we can get $b_{\mu\nu} = D_{[\nu}a_{\mu]}$. Therefore, if we can choose the gauge field (ρ_μ) so that the relation $D_\mu = a_\mu$ may hold good, we can compensate two kinds of arbitrariness appearing in (4.4) and (4.15) by our gauge field, as mentioned before.

By the way, owing to the Bianchi identity (Schouten, 1954), we can get the following fundamental equation of the gauge field:

$$\nabla_\mu F_{\lambda\kappa} + \nabla_\lambda F_{\kappa\mu} + \nabla_\kappa F_{\mu\lambda} = 0 \tag{4.16}$$

which should be regarded as a generalised form of Maxwell equation, as is emphasised by Bregman (1973) and Utiyama (1973).

5. *Conclusion*

In this paper we have considered physico-geometrically our non-local field, i.e. the (x, ψ)-field, by attaching weight to the non-vectorial property of the spinor (ψ). And we have also made clear some essential features underlying the concepts of metric and connection by introducing Weyl's gauge field and gauge invariance.

It might be said that our gauge field is more general than Weyl's and that it plays a role of a general material field instead of the electromagnetic potential. Therefore in future we should take note of the fundamental equations of the spin curvature tensors and should investigate the physical meaning of our gauge field in more detail.

As to the field theory, it is found that our field theory has close analogy to wave geometry. Therefore, we should also develop our geometrical theory of physical fields by taking wave geometry as a good model.

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